Research Activity

Antonio J. Di Scala

I am principally interested in connections, curvature and holonomy and their interaction with submanifolds. Submanifolds enters in the picture in several ways as the holonomy subbundles of a reduction of the holonomy group, as the orbits of the action of the holonomy group, etc. Since Elie Cartan's work the relation between symmetric spaces and holonomy is well-known. An important role are played by both the holonomy group associated to the Levi-Civita connection and that of the normal connection of a submanifold. Other holonomy groups are also the object of my study e.g. holonomy of the Chern connection, conformal holonomy, etc. I apply the above concepts to the study of special kind of submanifolds, for example the so called helix submanifolds. Kähler submanifolds are also studied in the above framework enriched by Calabi's Diastasis function. I take advantage of the two approaches to handle Hermitian Symmetric Spaces: the classical Elie Cartan's approach from Lie Theory and the Max Koecher's Jordan algebra viewpoint. I am also interested in the uniformization problem for compact or algebraic varieties since under natural assumptions the universal covering of such varieties are Hermitian symmetric spaces. Below you will find more details, examples and references about my research activity.

Research topics

1. Submanifolds and Holonomy
2. Kähler maps
3. Geometria Hermitiana
4. Dualità simplettica
5. Uniformizzazione
6. Sottovarietà elicoidali
7. Miscellaneous topics

1.1 Tensore di curvatura

1854 Riemann's Habilitationsvortrag:

Ueber die Hypothesen, welche der Geometrie zu Grunde liegen

... wenn also das Krümmungsmass in jedem Punkte in $n\frac{n-1}{2}$ Flächenrichtungen gegeben wird, so werden daraus die Massverhältnisse der Mannigfaltigkeit sich bestimmen lassen, wofern nur zwischen diesen Werthen keine identischen Relationen stattfinden, was in der That, allgemein zu reden, nicht der Fall ist.

... es reicht aber nach der frühern Untersuchung, um die Massverhältnisse zu bestimmen, hin zu wissen, dass es in jedem Punkte in $n\frac{n-1}{2}$ Flächenrichtungen, deren Krümmungsmasse von einander unabhängig sind, Null sei.

$$\mathrm{Q}(e_1, e_2) := \text{ curvatura sezionale Span}(e_1, e_2) \ .$$

Assertion [Spivak, Vol. II]. If M is n-dimensional and if $\mathrm{Q} = 0$ for $n\frac{n-1}{2}$ independent 2-dimensional subspaces of each M_q, then M is flat.

1.1 Controesempi

$\mathbf{R} : \Lambda^2(\mathbb{R}^3) \to \Lambda^2(\mathbb{R}^3)$, tensore di curvatura algebrico

$$\mathbf{R} := \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$\mathrm{Q}(e^{12}) = \mathrm{Q}(e^{23}) = \mathrm{Q}(e^{13}) = 0$

$S^2 \times \mathbb{RH}^2$

$\{e_1, e_2\}$ riferimento locale di S^2 e $\{a_1, a_2\}$ di $\mathbb{RH}^2$

$$f_1 = \frac{e_1+a_1}{\sqrt{2}}$$
$$f_2 = \frac{e_1-a_1}{\sqrt{2}}$$
$$f_3 = \frac{e_2+a_2}{\sqrt{2}}$$
$$f_4 = \frac{e_2-a_2}{\sqrt{2}}$$

$\mathrm{Q}(f^{12}) = \mathrm{Q}(f^{13}) = \mathrm{Q}(f^{14}) = \mathrm{Q}(f^{23}) = \mathrm{Q}(f^{24}) = \mathrm{Q}(f^{34}) = 0$

L'enseignement Mathématique t.47, 57-63, (2001).

Marcel Berger, A Panoramic View of Riemannian Geometry, Springer 2003. Page 214.

B. Riemann, On the Hypotheses Which Lie at the Bases of Geometry, Edited by J. Jost, 2016, Birkhäuser, Classical Texts in the Sciences. Page 87.

O. Darrigol, The mystery of Riemann's curvature, Historia Math. 42 (2015), no. 1, 47–83.

1.2 Olonomia normale

$M \subset \mathbb{C}^n$ sottovarietá non-degenere

$\nu(M) \to M$ fibrato normale, i.e. $\nu(M) \oplus TM = T\mathbb{C}^n$

$\nabla^\perp$ connessione normale in $\nu(M)$

$\Phi^\perp \subset SO(\nu(M))$ gruppo ristretto di olonomia di $\nabla^\perp$

Teorema [-]. Se $\Phi^\perp$ non agisce irriducibilmente allora M è un prodotto estrinseco, cioè $M = M_1 \times M_2 \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_1}$.

Teorema[-,Vittone] Se $\Phi^\perp$ agisce irreducibilmente ma non transitivamente allora

$$M \subset \mathcal{CM}_{smooth}$$

dove $\mathcal{CM} \subset \mathbb{C}^n$ è una varietà affine, un cosiddetto cono di Mok e $\dim(\mathcal{CM}) = \dim(M)$.

Advances in Mathematics (to appear).

Math. Ann. 351, no. 1, 187-214, (2011).

Mathematische Zeitschrift 261, 1-11, (2009).

Proc. London Math. Soc. (3) 89, 193-216, (2004).

l'Enseignement Mathematique 48, 23-50, (2002).

Mathematische Zeitschrift 235, 251-257, (2000).

1.3 Sottovarietà isoparametriche

$M \subset \mathbb{R}^n$ sottovarietá non-degenere

$\nu_0(M) \to M$ sottofibrato piatto dello spazio normale

$\eta_1, \cdots, \eta_k$ vettori principali di curvatura i.e. diagonalizzazione simultanea dei shape operators $A_\xi, \xi \in \nu_0(M)$

Teorema [-,Olmos]. Se M è completa e $\|\eta_j\| = cte_j$ allora M è isoparametrica. Falso se M non è completa.

Journal Für Die Reine und Angewandte Mathematik 574, 79-102, (2004).

Int. J. Geom. Methods Mod. Phys. 3, no. 5-6, 1019–1023, (2006).

Journal of Geometry 85, 32-34, (2006).

Proc. Amer. Math. Soc. 129, 3445-3446, (2001).

1.4 Sottovarietá omogenee minimali

$G \subset Iso(\mathbb{R}^n)$ gruppo di Lie connesso

$M = G.p \subset \mathbb{R}^n$ sottovarietá omogenea

α seconda forma fondamentale di M

H vettore curvatura media di M, cioè $H = \mathrm{tr}(\alpha)$

Teorema [-]. Se $H \equiv 0$ allora $\alpha \equiv 0$.

Bull. London Math. Soc. 35,825-827, (2003).

Lie Groups and Symmetric Spaces, AMS Translations–Series 2, Vol: 210. (2003).

Annals of Global Analysis and Geometry 21, p. 15-18, (2002).

Mathematische Zeitschrift 237 , p. 199-209, (2001).

1.5 Olonomia Lorentziana e Conforme

(M, g) varietá di Lorentz, cioè g ha segnatura $(n, 1)$

∇ connessione di Levi-Civita di g

Φ gruppo di olonomia ristretto di g

Teorema [Berger]. Se Φ agisce irriducibilmente allora $\Phi = SO_0(n, 1)$.

[Berard Bergery-Ikemakhen]:

è possibile fare una dimostrazione diretta?

Teorema [-,Olmos]. Se $G \subset SO(n, 1)$ è connesso e agisce irriducibilmente su $\mathbb{R}^{n+1}$ allora

$$G = SO_0(n, 1)$$

Corrigendum for "A geometric proof of the Karpelevich-Mostow theorem"

Bull. Lond. Math. Soc. 41, no. 4, 634–638, (2009).

Mathematische Zeitschrift 237 , p. 199-209, (2001).

Teorema [-,Leistner]. Se $G \subset SO(n,2)$ è connesso e agisce irriducibilmente su $\mathbb{R}^{n+2}$ allora G è coniugato a

1. $n \geq 1$: $SO_0(n,2)$,
2. $n = 2p$: $U(p,1), SU(p,1)$, o $SO(2) \cdot SO_0(p,1)$ se $p > 1$,
3. $n = 3$: $SO_0(1,2) \subset SO(3,2)$.

Differential Geom. Appl. 33 , 4-43, (2014).

ESI Lect. Math. Phys. 16, 629-651, (2010).

Israel Journal of Mathematics, (2009).

1.7 Nullity of complex submanifolds

$M \subset \mathbb{CP}^n$ complete complex submanifold

$\mathcal{N} = \{X \in TM : \alpha(X, \cdot) = 0\}$ nullity distribution of the second fundamental form α of M.

Theorem[Abe],[-,Olmos]. If $\dim(\mathcal{N}) > 0$ then M is totally geodesic, i.e. $\alpha \equiv 0$ or equivalent $M = \mathbb{CP}^m$ linearly embedded in $\mathbb{CP}^n$.

Proc. Amer. Math. Soc. 144 (2016), 1689-1695

Comment: If M is compact then due to Chow's theorem M is algebraic hence: Griffiths-Harris, Ann. Sci. Ecole Norm. Sup. (4) 12 (1979) :

ALGEBRAIC GEOMETRY AND LOCAL DIFFERENTIAL GEOMETRY 393

In particular, the points of intersection in (2.28) are necessarily singular points of M. From this we may draw the following global conclusion:

(2.29) *The only smooth projective variety having a degenerate Gauss mapping is* $\mathbb{P}^n$ *itself* ([36]).

([36]) Of course, (2.29) may be proved by global considerations, using e.g., a Chern class argument applied to the tangent bundle along a generic fibre of the Gauss mapping. Also, Alan Landman showed us a proof using some results of his on dual varieties.

2.1 Kähler maps

(M_1, ω_1), (M_2, ω_2) Kähler manifolds

$f : M_1 \to M_2$ è *Kähler map* se f olomorfo e $f^*\omega_2 = \omega_1$

Congettura[Clozel-Ullmo, *J. Reine Angew. Math. 558 (2003), 4783.*]:
A Kähler map

$$f : \mathbb{CH}^1 \to \mathbb{CH}^1 \times \mathbb{CH}^1$$

is totally geodesic.

FALSO:

$$f(z) := (\frac{z + \psi(z)}{\sqrt{2}}, \frac{z - \psi(z)}{\sqrt{2}})$$

dove $\psi(z) = -1 + \sqrt{1 + z^2}$ è un contraessempio.

Rapporto interno N.1, Politecnico di Torino, (2010).

See also Sui-Chung Ng, On holomorphic isometric embeddings of the unit disk into polydisks

2.1 La funzione Diastasi di Calabi

$D(p,q)$ Diastasi, cioè $D(p,\cdot)$ potenziale tale che

$$\left.\frac{\partial^k D(p,\cdot)}{\partial z^k}\right|_{z=p} = 0 \qquad k = 0, 1, \cdots$$

Teorema [-,Loi]. Sia (M,ω) uno spazio simmetrico Kähleriano. Se $M \to \ell^2(\mathbb{C})$ Kähler map allora M è un prodotto di spazi iperbolici e di uno spazio piatto.

Geom. Dedicata 125, 103–113 (2007)

Teorema [-,Ishi,Loi]. Sia (M,ω) uno spazio omogeneo Kähleriano. Se $M \to \ell^2(\mathbb{C})$ è Kähler map allora M è un prodotto di spazi iperbolici e di uno spazio piatto.

The Asian Journal of Mathematics, vol. 16, nro. 3, 479–488 (2012).

2.2 Parenti Kähleriani

(M, ω_M), (N, ω_N) due varietá di Kähler

M e N sono *parenti Kähleriani* se esiste (C, ω_C) e due Kähler maps

$$C \to M$$

$$C \to N$$

Teorema [-,Loi]. Spazi simmetrici di tipo diverso non sono parenti.

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) Vol. IX (2010), 495-501 .

http://arxiv.org/abs/1601.05907 (2016)

Comment: Il concetto di essere Parenti Kähleriani è collegato con il problema classico della esistenza di curve olomorfe in ipersuperfici reali di $\mathbb{C}^n$, e.g. curve olomorfe nel bordo di un dominio $D \subset \mathbb{C}^n$ e.g. J.P. D'Angelo.

3.1 Geometria Hermitiana

(M, g, J) almost-Kähler manifold, cioé

J non necessariamente integrable ma $\omega_J := g(J\cdot, \cdot)$ chiusa

∇^J connessione di Chern i.e. u.s.t. $\nabla^J J = \nabla^J g = \mathrm{Tor}^{1,1}(\nabla^J) = 0$

R^J curvatura di Chern

Teorema [-,Vezzoni]. Se R^J verifica Bianchi allora J è integrable.

Proceedings of Edinburgh Math. Society (2009).

(M, g, J) quasi-Kähler manifold, cioé

J non necessariamente integrable ma $\omega_J := g(J\cdot, \cdot)$ è (1,2)-chiusa i.e. $\overline{\partial}\omega_J \equiv 0$

Teorema [-,Vezzoni]. Se M compatta e $Hol(\nabla^J) = \{e\}$ allora M 2-step nilmanifold.

Math. Z. 271, no. 1-2, 95-108, (2012).

Ann. Global Anal. Geom. 40, no. 1, 21-45, (2011).

Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 11 , no. 1, 41-60, (2012).

3.2 Geometria Hermitiana

$\mathrm{T} = \mathbb{C}/\Lambda$ toro

$(\mathbb{R}^4, g_0, J)$ OAC structure

Teorema [-,Vezzoni]. Esiste OAC in $\mathbb{R}^4$ e $f : \mathrm{T} \to \mathbb{R}^4$ isometrico e olomorfo.

Q J Math 61 (4): 401-405 (2010).

M^{2n} varietá complessa compatta

Teorema [-,Zuddas]. Esiste AC structure in $\mathbb{R}^{6n}$ e $f : M \to \mathbb{R}^{6n}$ olomorfo.

J. Geom. Phys. 61, no. 10, 1928-1931, (2011).

3.3 Geometria Hermitiana

[Apostolov-Draghici]: $\mathbb{CH}^2$ does admit a (locally defined) strictly almost-Kähler structure J which is compatible with the canonical metric and with the standard orientation ? Same question for $\mathbb{CH}^1 \times \mathbb{CH}^1$.

The curvature and the integrability of almost-Kahler manifolds: a survey.

Symplectic and Contact Topology: Interactions and Perspectives

Fields Institute Communications Series, AMS, 2003.

arXiv:math/0302152v1 [math.DG]

Teorema [-,Nagy]. Sia (M, g, J) una superficie Kähler-Einstein e sia J' OAC. Se $g(J'\cdot, \cdot)$ è chiusa allora J' è integrable.

C. R. Math. Acad. Sci. Paris 348, no. 7-8, 423–425, (2010).

3.4 Non Kähler structures on $\mathbb{R}^4$

In 1953 Calabi-Eckmann constructed complex structures J on $\mathbb{R}^{2n}$ where $n \geq 3$ which are non Kähler i.e. there is no symplectic form ω such that $\omega(J\cdot,\cdot)$ is a Riemannian metric. As they wrote the motivation comes from some questions of S. Bochner:

> The three properties we have proved about the cells $\mathbf{E}_{p,q}(q > 0)$ of complex dimension $p + q + 1$ answer negatively, at least in the case of complex dimension $\geqq 3$, some of the questions raised by Bochner, [1], regarding the possibility of uniformization of complex manifolds homeomorphic to cells.

They said nothing about complex dimension 2, that is to say, about the existence of non-Kähler complex structures on $\mathbb{R}^4$.

Teorema [-,Kasuya, Zuddas]. There are Calabi-Eckmann type complex structures on $\mathbb{R}^4$.

To appear in Geometry and Topology.
http://arxiv.org/abs/1501.06097 (2015)

http://arxiv.org/abs/1511.08471 (2016)

4.1 Dualità simplettica

$(\mathbb{C}, \omega_0, \omega_{\mathrm{FS}})$

$(\Delta, \omega_{\mathrm{hyp}}, \omega_0)$

$\Phi : \Delta \to \mathbb{C} \subset \Delta^* = \mathbb{CP}^1$

$\Phi(z) := \frac{z}{\sqrt{1-|z|^2}}$

Φ è *dualità simplettica*, cioè $\begin{cases} \Phi^*\omega_0 = \omega_{\mathrm{hyp}}, \\ \Phi^*\omega_{\mathrm{FS}} = \omega_0. \end{cases}$

Teorema [-,Loi]. Sia D un dominio simmetrico e sia $\mathrm{B}(z, w)$ il suo nucleo di Bergman. Sia $\Phi : D \to D^*$ definito da

$$\Phi(z) := \mathrm{B}(z, z)^{-\frac{1}{4}} z\,.$$

Allora Φ è dualitá simplettica.

Advances in Mathematics 217, No. 5, (2008).

Transformation Groups 13, No. 2, 283-304 (2008).

Monatshefte für Mathematik 160, No. 4, 403-420, (2010)

5.1 Uniformizzazione

Theorem [Uniformization theorem of Koebe and Poincaré]. Let C be a smooth (connected) compact complex curve of genus g, and let $\widetilde{C}$ be its universal cover. Then

$$\widetilde{C} \cong \begin{cases} \mathbb{P}^1 & \text{if } g = 0 \\ \mathbb{C} & \text{if } g = 1 \\ \mathbb{H} & \text{if } g \geq 2 \end{cases}$$

Theorem[Catanese-Franciosi, On varieties whose universal cover is a product of curves, Contemporary Mathematics 496, 2009.]
Let X be a compact complex manifold of dimension $n \leq 3$ and let $\widetilde{X}$ be its universal cover. Then the following two conditions:

(1) X admits a semi special tensor ω i.e. a non zero section of the sheaf $S^n(\Omega^1_X)(-K_X) \otimes \eta$, where η invertible sheaf such that $\eta^2 = \mathcal{O}_X$;

(2) K_X is ample

hold if and only if $\widetilde{X} \cong \mathbb{H}^2$ or $\widetilde{X} \cong \mathbb{H}^3$.

Falso for $n \geq 4$.

5.1 Uniformizzazione

Theorem[Catanese,-]. Let X be a compact complex manifold of dimension n and let $\widetilde{X}$ be its universal cover.

Then the universal cover $\widetilde{X}$ is the polydisk $\mathbb{H}^n$ if and only if

(1) K_X is ample

(2) X admits a semi special tensor ω of relative degree n with reduced divisor;

Comment 1: The existence of more general tensor, the so called zero slope tensors, are used to state and prove general necessary and sufficient conditions under which a compact complex manifold is uniformized by a bounded symmetric domain of tube type.

Comment 2: By using even more general curvature like tensors we also give necessary and sufficient conditions for uniformization by bounded symmetric domains without balls.

Math. Ann. 356 , no. 2, 419–438, (2013).

Adv. Math. 257 , 567–580, (2014).

6.1 Sottovarietà elicoidali

$d \in \mathbb{R}^n$, $\|d\| = 1$

$M \subset \mathbb{R}^n$ sottovarietá

M è *elica* se l'angolo $\angle(T_pM, d)$ non dipende da $p \in M$.

[Cermelli,-]. Classificazione delle eliche di $\mathbb{R}^3$ e applicazioni alla teoria dei cristali liquidi.

Philosophical Magazine vol. 87, pp. 1871-1888 (2007)

[-,Ruiz-Hernandez]. Studio e classificazione delle sottovarieta' elicoidali di $\mathbb{R}^n$.

Bol. Soc. Mat. Mex. (3) 22 , no. 1, 229-250, (2016).

Beitr. Algebra Geom. 56 , no. 2, 551-573, (2015).

Bull. Braz. Math. Soc. (N.S.) 46 , no. 1, 105-138, (2015).

Geom. Dedicata 162 , 153-176, (2013).

Monatsh. Math. 168 , no. 2, 183-189, (2012).

Kodai Math. Journal. (2010).

Monatshefte für Mathematik 157, No. 3, 205-215, (2009).

Abh. Math. Semin. Univ. Hambg. 79: 37-46, (2009).

7.1 Miscellaneous topics

[I.Bauer & F. Catanese,-] *Higher dimensional Lemniscates: the geometry of r particles in n-space with logarithmic potentials*, to appear in Annali SNS Pisa.

[R. Camporesi,-] *A generalization of a theorem of Mammana*, Colloq. Math. 122 , no. 2, 215-223, (2011)

[-,M. Sombra] *Intrinsic palindromes*, Fibonacci Quart. 42 , no. 1, 7681, (2004). , see World of numbers

[G. Casnati,-,R. Notari] *Workshop on Hodge theory and algebraic geometry, Trento 2009*, Rend. Semin. Mat. Univ. Politec. Torino 68 , no. 3, 199-206, (2010)

A flat but non-smoothly ruled surface, PORTO

El Teorema de Dehn, Revista de Educacion de la Union Matematica Argentina 16, Nro. 2, p. 22-35, (2001)

Regla de L'Hopital para sucesiones, Revista de Educacion de la Union Matematica Argentina 13, Nro. 1, p. 15-20, (1998)